J Glob Optim (2009) 45:551–563
DOI 10.1007/s10898-008-9389-4

Set containment characterization for quasiconvex programming

Satoshi Suzuki · Daishi Kuroiwa

Received: 12 January 2008 / Accepted: 6 December 2008 / Published online: 24 December 2008

Abstract Dual characterizations of containment of a convex set, defined by quasiconvex constraints, in a convex set, and in a reverse convex set, defined by a quasiconvex constraint, are provided. Notions of quasiconjugate for quasiconvex functions, H-quasiconjugate and R-quasiconjugate, play important roles to derive characterizations of the set containments.

Keywords Set containment · Quasiconvex constraints · Quasiconjugate function · Reverse-convex set

1 Introduction

Classification is one of the basic problems in data mining which addresses the question of how best to use historical data to improve the process of making decisions and to discover regularities. Motivated by general nonpolyhedral knowledge-based data classification, the containment problem which consists of characterizing the inclusion $A \subset B$, where $A = \{x \in \mathbb{R}^n \mid f_i(x) \leq 0, i \in I\}$ and $B = \{x \in \mathbb{R}^n \mid h_j(x) \leq 0, j \in J\}$, was studied by many researchers. The first characterizations were given by Mangasarian [5] for linear systems and for systems involving differentiable convex functions, and key to this approach was Farkas' Lemma and the duality theorems of convex programming, respectively. Jeyakumar [4] established excellent dual characterizations of the set containment, assuming the convexity of $f_i, i \in I$, and the convexity (the concavity) of $h_j, j \in J$, so that A is a closed convex set and B is a closed convex set (a reverse convex set, respectively). Also, Goberna and Rodríguez [1] established characterizations of the set containment for linear systems containing strict inequalities and weak inequalities as well as equalities, and, Goberna et al.

S. Suzuki (✉) · D. Kuroiwa
Interdisciplinary Graduate School of Science and Engineering, Shimane University, Matsue, Japan
e-mail: suzuki@math.shimane-u.ac.jp

S. Suzuki · D. Kuroiwa
Interdisciplinary Faculty of Science and Engineering, Shimane University, Matsue, Japan

[2] characterized set containments with convex inequalities, which can be either weak or strict inequalities.

It is well known that the Fenchel conjugate provides dual problems of convex minimization problems. In a similar way, different notions of conjugate for quasiconvex functions have been introduced in order to obtain dual problems of quasiconvex minimization problems. For example, the λ-quasiconjugate ($\lambda \in \mathbb{R}$), defined by Greenberg and Pierskalla [3], plays in quasiconvex optimization and in the theory of surrogate duality the same role played by the Fenchel conjugate in convex optimization and Lagrangian duality. But λ-quasiconjugate involves an extra parameter that many authors have tried to eliminate. Thach [6,7] established two dualities without the extra parameter for a general quasiconvex minimization (maximization) problem, by using the concepts of quasiconjugate and R-quasiconjugate, which are similar to 1 and -1-quasiconjugate.

Motivated by these works, we establish in this paper dual characterizations of the set containment, assuming the quasiconvexity of $f_i, i \in I$, the linearity or quasiconcavity of $h_j, j \in J$, that A is defined by strict inequalities and B by both types of inequalities, so that A is convex whereas B is either convex or reverse convex. The dual characterizations are provided in terms of level sets of quasiconjugate and R-quasiconjugate of quasiconvex functions.

2 Notation and preliminaries

Throughout this paper, let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. Remember that f is said to be quasiconvex if, for all $x_1, x_2 \in \mathbb{R}^n$ and $\alpha \in (0, 1)$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Define

$$L(f, \diamond, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \diamond \alpha\}$$

for any $\alpha \in \mathbb{R}$. Symbol $\diamond$ represents any binary relation. Then f is quasiconvex if and only if for any $\alpha \in \mathbb{R}$,

$$L(f, \leq, \alpha) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$$

is a convex set, or equivalently, for any $\alpha \in \mathbb{R}$,

$$L(f, <, \alpha) = \{x \in \mathbb{R}^n \mid f(x) < \alpha\}$$

is a convex set. We know that any convex function is quasiconvex. The converse is not true.

Definition 1 A subset S of $\mathbb{R}^n$ is said to be evenly convex if it is the intersection of some family of open halfspaces.

Note that the whole space and the empty set are evenly convex. Also, any open convex set and any closed convex set are evenly convex. Clearly, every evenly convex set is convex.

Definition 2 A function f is said to be evenly quasiconvex if $L(f, \leq, \alpha)$ is evenly convex for all $\alpha \in \mathbb{R}$.

Definition 3 A function f is said to be strictly evenly quasiconvex if $L(f, <, \alpha)$ is evenly convex for all $\alpha \in \mathbb{R}$.

Clearly, every evenly quasiconvex function is quasiconvex, every lower semicontinuous (lsc) quasiconvex function is evenly quasiconvex, and every upper semicontinuous (usc) quasiconvex function is strictly evenly quasiconvex. It is easy to show that every strictly evenly quasiconvex function is evenly quasiconvex, but the converse is not generally true.

Example 1 Consider the function

$$f(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_1 \geq 0 \text{ and } x_2 \leq x_1, \\ +\infty & \text{if} \quad x_2 < 0 \text{ or } x_1 < 0, \\ 1 & \text{if} \quad x_1 = 0 \text{ and } x_2 > 0, \\ 1 - \frac{x_1}{x_2} & \text{if} \quad x_1 > 0 \text{ and } x_2 > x_1. \end{cases}$$

The function f is evenly quasiconvex, but not strictly evenly quasiconvex because $L(f, <, \alpha)$ is not evenly convex when $\alpha \in (0, 1]$.

3 *H*-quasiconjugacy and *H*-quasiconvexity

In this paper, we use two concepts of quasiconjugacy, due to Thach [6,7], that we distinguish by the prefixes "H-" and "R-".

Definition 4 (*[6]*) H-quasiconjugate of f is the function $f^H : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that

$$f^H(\xi) = \begin{cases} -\inf\{f(x) \mid \langle \xi, x \rangle \geq 1\} & \text{if} \quad \xi \neq 0 \\ -\sup\{f(x) \mid x \in \mathbb{R}^n\} & \text{if} \quad \xi = 0. \end{cases}$$

The H-quasiconjugate of f^H, say f^{HH}, is called the H-biquasiconjugate of f.

Clearly, $f^H(0) \leq f^H(x)$ for all $x \in \mathbb{R}^n \backslash \{0\}$, $f^{HH} \leq f$ on $\mathbb{R}^n \backslash \{0\}$, $f^H \leq g^H$ on $\mathbb{R}^n \backslash \{0\}$ when $f \geq g$ on $\mathbb{R}^n \backslash \{0\}$, and $f^H = g^H$ on $\mathbb{R}^n \backslash \{0\}$ when $f = g$ on $\mathbb{R}^n \backslash \{0\}$. Then we have the following inequalities:

Proposition 1 (i) $\sup_{x \in \mathbb{R}^n} f^H(x) \leq -\inf_{x \in \mathbb{R}^n} f(x)$*;*
(ii) $-\sup_{x \in \mathbb{R}^n} f(x) \leq \inf_{x \in \mathbb{R}^n \backslash \{0\}} f^H(x)$.

Definition 5 A subset S of $\mathbb{R}^n$ is said to be H-evenly convex if it is the intersection of some family of open halfspaces, and each open halfspace containing 0.

Note that the whole space and the empty set are H-evenly convex. Also it is clear that a nonempty subset S of $\mathbb{R}^n$ is H-evenly convex if and only if S is an evenly convex set which contains 0.

Definition 6 A function f is said to be H-evenly quasiconvex if $L(f, \leq, \alpha)$ is H-evenly convex for all $\alpha \in \mathbb{R}$.

Definition 7 A function f is said to be strictly H-evenly quasiconvex if $L(f, <, \alpha)$ is H-evenly convex for all $\alpha \in \mathbb{R}$.

It is clear that every strictly H-evenly quasiconvex function is H-evenly quasiconvex, but the converse implication is not true, as Example 1 shows. Also a function f is H-evenly quasiconvex if and only if f is evenly quasiconvex and $f(0) = \inf_{x \in \mathbb{R}^n} f(x)$. Moreover we can check that f^H is H-evenly quasiconvex, in a similar way of [7]. We can also see that the equality $f^{HH}(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}$ holds in [6]. From this equality we characterize the identity $f = f^{HH}$ in the next theorem, whose proof will be given after Proposition 3.

Theorem 1 *The following properties are satisfied:*

(i) $f = f^{HH}$ *on* $\mathbb{R}^n \backslash \{0\}$ *if* f *is* H*-evenly quasiconvex.*
(ii) $f = f^{HH}$ *if and only if* f *is* H*-evenly quasiconvex and*

$$f(0) = \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}.$$

Definition 8 We say that f achieves the maximum value at infinity if $f(x_k) \to \sup\{f(x) \mid x \in \mathbb{R}^n\}$ for any sequence $\{x_k\}$ with $\|x_k\| \to +\infty$.

Definition 9 We say that f achieves the minimum value at the origin if $f(x_k) \to \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}$ for any sequence $\{x_k\} \subset \mathbb{R}^n \backslash \{0\}$ with $x_k \to 0$.

Let Γ^∞ and γ^0 be the set of all functions that achieve the maximum value at infinity, and the set of all functions that achieve the minimum value at the origin, respectively; that is,

$$\Gamma^\infty = \{g : \mathbb{R}^n \to \overline{\mathbb{R}} \mid g \text{ achieves the maximum value at infinity}\},$$
$$\gamma^0 = \{g : \mathbb{R}^n \to \overline{\mathbb{R}} \mid g \text{ achieves the minimum value at the origin}\}.$$

We denote by X^c the complement of $X \subset \mathbb{R}^n$ and by $B(z, r)$ the open ball centered at $z \in \mathbb{R}^n$ with radius $r > 0$.

Proposition 2 *The following properties are satisfied:*

(i) $f \in \Gamma^\infty$ *if and only if for any* $M < \sup\{f(x) \mid x \in \mathbb{R}^n\}$ *there exists* $\delta > 0$ *such that*

$$B(0, \delta)^c \subset L(f, \geq, M).$$

(ii) $f \in \gamma^0$ *if and only if for any* $m > \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}$ *there exists* $\delta > 0$ *such that*

$$B(0, \delta) \backslash \{0\} \subset L(f, <, m).$$

Proof We only show (ii); we can show (i) in the similar way. Assume that f achieves the minimum value at the origin and there exists $m_0 > \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}$ such that for any $\delta > 0$, there exists $x \in B(0, \delta) \backslash \{0\}$ such that $f(x) \geq m_0$. Then we can choose a sequence $\{x_k\} \subset \mathbb{R}^n \backslash \{0\}$ converging to 0. This contradicts that f achieves the minimum value at the origin. Conversely, assume that for any $m > \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}$, there exists $\delta > 0$ such that $B(0, \delta) \backslash \{0\} \subset L(f, <, m)$. If $\{x_k\} \subset \mathbb{R}^n \backslash \{0\}$ converges to 0, then there exists $K \in \mathbb{N}$ such that $\|x_k\| < \delta$ for any $k \geq K$. This shows that $\inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\} \leq f(x_k) < m$ for any $k \geq K$. This shows $f(x_k) \to \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}$. □

According to [6], f is usc then f^H is lsc, and if a function $f \in \Gamma^\infty$ is lsc, then f^H is usc.

Theorem 2 *The following properties are satisfied:*

(i) *If* $f \in \gamma^0$ *then* $f^H \in \Gamma^\infty$*;*
(ii) *If* $f \in \Gamma^\infty$ *then* $f^H \in \gamma^0$.

Proof (i) Let $f \in \gamma^0$ and $\{x_k\} \subset \mathbb{R}^n$ be a sequence satisfying $\|x_k\| \to +\infty$. By using (ii) of Proposition 2, for any $m > \inf\{f(x) \mid x \in \mathbb{R}^n \backslash \{0\}\}$, there exists $\delta > 0$ such that

$$B(0, \delta) \backslash \{0\} \subset L(f, <, m).$$

Since $\|x_k\| \to +\infty$, we can find an integer K such that for any $k \geq K$, $\frac{x_k}{\|x_k\|^2} \in B(0,\delta)\backslash\{0\}$. Also since $\left\langle x_k, \frac{x_k}{\|x_k\|^2} \right\rangle = 1$, by using (i) of Proposition 1, we can show that

$$\inf\{f(x) \mid x \in \mathbb{R}^n\backslash\{0\}\} \leq -\sup_{x\in\mathbb{R}^n} f^H(x) \leq -f^H(x_k) \leq f\left(\frac{x_k}{\|x_k\|^2}\right) < m.$$

This shows that $f^H(x_k) \to \sup\{f^H(x) \mid x \in \mathbb{R}^n\}$, and then $f^H \in \Gamma^\infty$.

(ii) Let $f \in \Gamma^\infty$ and $\{x_k\} \subset \mathbb{R}^n\backslash\{0\}$ be a sequence satisfying $x_k \to 0$. For any $M < \sup\{f(x) \mid x \in \mathbb{R}^n\}$, there exists $\delta > 0$ such that

$$B(0,\delta)^c \subset L(f, \geq, M),$$

by Proposition 2 (i). Since $x_k \to 0$, we can find an integer K such that for any $k \geq K$,

$$\{x \mid \langle x_k, x\rangle \geq 1\} \subset B(0,\delta)^c,$$

that is,

$$\langle x_k, x\rangle \geq 1 \Longrightarrow f(x) \geq M.$$

From this and by using (ii) of Proposition 1, we have

$$\sup\{f(x) \mid x \in \mathbb{R}^n\} \geq -\inf_{x\in\mathbb{R}^n\backslash\{0\}} f^H(x) \geq -f^H(x_k) \geq M,$$

for any $k \geq K$. This shows that $f^H(x_k) \to \inf\{f^H(x) \mid x \in \mathbb{R}^n\backslash\{0\}\}$, and then $f^H \in \gamma^0$. □

4 Level set of biquasiconjugate

Given a set $S \subset \mathbb{R}^n$, we shall denote by int$S$, cl$S$, co$S$ and cone$S$ the interior, the closure, the convex hull, and the conical hull generated by S, respectively. The evenly convex hull of S, denoted by ec$S$, is the smallest evenly convex set which contains S (i.e., it is the intersection of all open halfspaces which contain S). The H-evenly convex hull of S, denoted by Hec$S$, is the smallest H-evenly convex set which contains S. Note that co$S \subset$ ec$S \subset$ clco$S$, and these differences are slight because clco$S$ = clec$S$. Moreover if S is nonempty, then Hec$S$ = ec$(S \cup \{0\})$.

Proposition 3 *Let S be a nonempty subset of $\mathbb{R}^n$.*

(i) *An element $x \in \mathbb{R}^n$ satisfies $x \notin$ ec$S$ if and only if there exists $a \in \mathbb{R}^n\backslash\{0\}$ and $\alpha \in \mathbb{R}$ such that, for all $y \in S$,*

$$\langle a, x\rangle \geq \alpha > \langle a, y\rangle .$$

(ii) *An element $x \in \mathbb{R}^n$ satisfies $x \notin$ Hec$S$ if and only if there exists $a \in \mathbb{R}^n\backslash\{0\}$ such that, for all $y \in S$,*

$$\langle a, x\rangle \geq 1 > \langle a, y\rangle .$$

Proof (i) The proof is easy since ec$S$ is equal to the intersection of the family of all open halfspaces which contain S. (ii) By using (i) above, $x \notin$ Hec$S$ = ec$(S \cup \{0\})$ if and only if there exists $a \in \mathbb{R}^n\backslash\{0\}$ and $\alpha \in \mathbb{R}$ such that, for all $y \in S \cup \{0\}$,

$$\langle a, x\rangle \geq \alpha > \langle a, y\rangle .$$

Taking $y = 0$, we have $\alpha > 0$, and hence, for all $y \in S$,

$$\left\langle \frac{a}{\alpha}, x \right\rangle \geq 1 > \left\langle \frac{a}{\alpha}, y \right\rangle.$$

The converse is clear. □

Now we prove Theorem 1 by using Proposition 3.

Proof of Theorem 1 We may prove (i). It is clear that $f(x) \geq f^{HH}(x)$ for all $x \in \mathbb{R}^n \backslash \{0\}$. Assume that there exists $x_0 \in \mathbb{R}^n \backslash \{0\}$ such that $f(x_0) > f^{HH}(x_0)$. We can choose $\alpha \in \mathbb{R}$ satisfying

$$f(x_0) > \alpha > f^{HH}(x_0),$$

and then $x_0 \notin L(f, \leq, \alpha)$. Since $L(f, \leq, \alpha)$ is H-evenly convex, there exists $v \in \mathbb{R}^n \backslash \{0\}$ such that $\langle v, x \rangle \geq 1 > \langle v, y \rangle$ for all $y \in L(f, \leq, \alpha)$ by using Proposition 3. This shows that $f^H(v) \leq -\alpha$. Therefore

$$f^{HH}(x_0) = -\inf\{f^H(u) \mid \langle u, x_0 \rangle \geq 1\} \geq -f^H(v) \geq \alpha.$$

This is a contradiction. □

Next we show properties of level sets of H-biquasiconjugate. For this purpose, we prove the following proposition.

Proposition 4 *Let $\alpha, \beta \in \mathbb{R}$, and $v \in \mathbb{R}^n \backslash \{0\}$. If $f \in \Gamma^\infty$ and f is lsc, then the following two conditions are equivalent:*

(i) $L(f, \leq, \beta) \subset \{x \mid \langle v, x \rangle < \alpha\}$,
(ii) $\exists \varepsilon > 0$ *s.t.* $L(f, <, \beta + \varepsilon) \subset \{x \mid \langle v, x \rangle < \alpha\}$.

Proof We show that condition (i) implies condition (ii). Assume that $L(f, \leq, \beta) \subset \{x \mid \langle v, x \rangle < \alpha\}$ and for all $\varepsilon > 0$ there exists $x \in L(f, <, \beta + \varepsilon)$ such that $\langle v, x \rangle \geq \alpha$, then we can choose a sequence $\{x_k\} \subset \mathbb{R}^n$ such that for all $k \in \mathbb{N}$, $\beta < f(x_k) < \beta + \frac{1}{k}$ and $\langle v, x_k \rangle \geq \alpha$, and we have $f(x_k) \to \beta < f(x_1) \leq \sup\{f(x) \mid x \in \mathbb{R}^n\}$. If $\|x_k\| \to +\infty$, then $f(x_k) \to \sup\{f(x) \mid x \in \mathbb{R}^n\}$ since $f \in \Gamma^\infty$. This is a contradiction. Hence $\{x_k\}$ is bounded, and we can choose a subsequence $\{x_{k_i}\}$ and $x_0 \in \mathbb{R}^n$ such that $x_{k_i} \to x_0$. Clearly $\langle v, x_0 \rangle \geq \alpha$, but $x_0 \in L(f, \leq, \beta)$ since $f(x_0) \leq \liminf_{i \to \infty} f(x_{k_i}) = \beta$. This is a contradiction. The converse implication is obvious. □

Now we can give results on the level sets of the H-biquasiconjugate.

Theorem 3 *The following properties are satisfied:*

(i) $L(f, \leq, \alpha) \backslash \{0\} \subset L(f^{HH}, \leq, \alpha)$;
(ii) $L(f, <, \alpha) \backslash \{0\} \subset L(f^{HH}, <, \alpha)$;
(iii) $\mathrm{Hec}L(f, \leq, \alpha) \subset L(f^{HH}, \leq, \alpha)$;
(iv) $L(f^{HH}, <, \alpha) \subset \mathrm{Hec}L(f, <, \alpha)$;
(v) *If $f \in \Gamma^\infty$ and f is lsc, then*

$$\mathrm{Hec}L(f, \leq, \alpha) = L(f^{HH}, \leq, \alpha) = \bigcap_{\varepsilon > 0} \mathrm{Hec}L(f, <, \alpha + \varepsilon).$$

Proof (i), (ii) and (iii) are obvious. At first, we show (iv). Assume that $x(\neq 0) \notin \mathrm{Hec}\, L(f, <, \alpha)$. By using Proposition 3, there exists $a \in \mathbb{R}^n \backslash \{0\}$ such that $\langle a, x\rangle \geq 1 > \langle a, y\rangle$ for all $y \in L(f, <, \alpha)$. Then

$$f^{HH}(x) = -\inf\{f^H(v) \mid \langle v, x\rangle \geq 1\} \geq -f^H(a) = \inf\{f(y) \mid \langle a, y\rangle \geq 1\} \geq \alpha.$$

Therefore $x \notin L(f^{HH}, <, \alpha)$. If $L(f^{HH}, <, \alpha)$ contains 0, then $L(f, <, \alpha)$ is not empty, and hence $\mathrm{Hec}L(f, <, \alpha)$ contains 0.

Next we show (v).

$$\mathrm{Hec}L(f, \leq, \alpha) \subset L(f^{HH}, \leq, \alpha) \subset \bigcap_{\varepsilon>0} \mathrm{Hec}L(f, <, \alpha + \varepsilon)$$

is obviously. We assume that $x \notin \mathrm{Hec}L(f, \leq, \alpha)$. By using Proposition 3, there exists $a \in \mathbb{R}^n \backslash \{0\}$ such that $\langle a, x\rangle \geq 1 > \langle a, y\rangle$ for all $y \in L(f, \leq, \alpha)$, then we have $L(f, \leq, \alpha) \subset \{y \mid \langle a, y\rangle < 1\}$. By using Proposition 4, there exists $\varepsilon_0 > 0$ such that $\langle a, x\rangle \geq 1 > \langle a, y\rangle$ for all $y \in L(f, <, \alpha + \varepsilon_0)$. By using Proposition 3 again, we have $x \notin \bigcap_{\varepsilon>0} \mathrm{Hec}L(f, <, \alpha + \varepsilon)$, and consequently

$$\bigcap_{\varepsilon>0} \mathrm{Hec}L(f, <, \alpha + \varepsilon) \subset \mathrm{Hec}L(f, \leq, \alpha).$$

This completes the proof. □

5 Containment of a convex set in an open halfspace

In this section, we present a characterization of the containment of a convex set, defined by quasiconvex constraints, in an open halfspace. For this purpose we show a result concerning the H-quasiconjugate of the sup-function, which plays an important role in this paper. We start with a result on the containment for the case $|I| = 1$.

Theorem 4 *Let* $v \in \mathbb{R}^n \backslash \{0\}$, $\alpha \in (0, \infty)$ *and* $\beta \in \mathbb{R}$. *Then*

$$L(f, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\} \Longleftrightarrow \frac{v}{\alpha} \in L(f^H, \leq, -\beta).$$

Proof Assume that $L(f, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\}$, i.e., that $f(x) < \beta$ implies $\langle v, x\rangle < \alpha$ or, equivalently, that $\left\langle \frac{v}{\alpha}, x\right\rangle \geq 1$ implies $f(x) \geq \beta$. This shows that

$$f^H\left(\frac{v}{\alpha}\right) = -\inf\left\{f(x) \,\middle|\, \left\langle \frac{v}{\alpha}, x\right\rangle \geq 1\right\} \leq -\beta.$$

Conversely, if $f^H(\frac{v}{\alpha}) \leq -\beta$, then $\inf\{f(x) \mid \left\langle \frac{v}{\alpha}, x\right\rangle \geq 1\} \geq \beta$. Therefore the inequality $\left\langle \frac{v}{\alpha}, x\right\rangle \geq 1$ implies $f(x) \geq \beta$, i.e., $f(x) < \beta$ implies $\langle v, x\rangle < \alpha$. Thus $L(f, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\}$. □

The theorem is valid when the constraint function is unique. Substituting $\sup_{i\in I} f_i$ into f, we have

$$L(\sup_{i\in I} f_i, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\} \Longleftrightarrow \frac{v}{\alpha} \in L((\sup_{i\in I} f_i)^H, \leq, -\beta),$$

for $v \in \mathbb{R}^n \backslash \{0\}$, $\alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$. We know that

(i) $(\inf_{i\in I} f_i)^H = (\sup_{i\in I} f_i^H)$ on $\mathbb{R}^n \backslash \{0\}$, and

(ii) If f is H-evenly quasiconvex, then $f^{HH} = f$ on $\mathbb{R}^n \backslash \{0\}$.

When every f_i is H-evenly quasiconvex, by substituting f_i^H into (i) we have

$$(\inf_{i \in I} f_i^H)^{HH} = (\sup_{i \in I} f_i^{HH})^H = (\sup_{i \in I} f_i)^H$$

on $\mathbb{R}^n \backslash \{0\}$. However, the H-quasiconvexity assumption is too strong because it assures $f_i(0) \leq f_i(x)$ for all $x \in \mathbb{R}^n$ and $i \in I$. The assumption of the next theorem is weaker than the previous one and guarantees that $(\inf_{i \in I} f_i^H)^{HH} = (\sup_{i \in I} f_i)^H$ on $\mathbb{R}^n \backslash \{0\}$.

Theorem 5 *Let I be an arbitrary index set, and f_i be a evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for all $i \in I$. If the condition*

(A1) $\sup_{i \in I} f_i(x) > \sup_{i \in I} f_i(0)$ *for all* $x \in \mathbb{R}^n \backslash \{0\}$,

is satisfied, then

$$(\sup_{i \in I} f_i)^H(v) = (\inf_{i \in I} f_i^H)^{HH}(v) \text{ for all } v \in \mathbb{R}^n \backslash \{0\}.$$

Proof In general, the following equality about H-quasiconjugate of inf-function is satisfied: for all $v \in \mathbb{R}^n \backslash \{0\}$,

$$(\inf_{i \in I} f_i)^H(v) = \sup_{i \in I} f_i^H(v),$$

see for example [6]. Then, for all $x \in \mathbb{R}^n \backslash \{0\}$, we have

$$(\inf_{i \in I} f_i^H)^H(x) = \sup_{i \in I} f_i^{HH}(x) \leq \sup_{i \in I} f_i(x),$$

hence,for all $v \in \mathbb{R}^n \backslash \{0\}$, we have

$$(\inf_{i \in I} f_i^H)^{HH}(v) \geq (\sup_{i \in I} f_i)^H(v).$$

If the equality does not hold in the above inequality, then there exists $v \in \mathbb{R}^n \backslash \{0\}$ such that $(\sup_{i \in I} f_i)^H(v) < (\inf_{i \in I} f_i^H)^{HH}(v)$, and hence, there exists $\alpha \in \mathbb{R}$ and $x' \in \mathbb{R}^n$ such that $\langle v, x' \rangle \geq 1$ and

$$(\sup_{i \in I} f_i)^H(v) < \alpha < \inf_{\langle w, x' \rangle \geq 1} \inf_{i \in I} f_i^H(w).$$

From $x' \neq 0$ and the assumption, we have $\sup_{i \in I} f_i(x') > \sup_{i \in I} f_i(0)$, and put $\varepsilon' = (\sup_{i \in I} f_i(x') - \sup_{i \in I} f_i(0))/2 > 0$. For all $\varepsilon \in (0, \varepsilon')$, there exists $i_0 \in I$ such that

$$f_{i_0}(x') > \sup_{i \in I} f_i(x') - \varepsilon > \sup_{i \in I} f_i(0) \geq f_{i_0}(0).$$

Since $L(f_{i_0}, \leq, \sup_{i \in I} f_i(x') - \varepsilon)$ does not contain x', contains 0, and it is evenly convex, there exists $a \in \mathbb{R}^n \backslash \{0\}$ such that for all $x \in L(f_{i_0}, \leq, \sup_{i \in I} f_i(x') - \varepsilon)$,

$$\langle a, x' \rangle \geq 1 > \langle a, x \rangle .$$

Therefore,

$$\alpha < \inf_{\langle w, x' \rangle \geq 1} \inf_{i \in I} f_i^H(w) \leq f_{i_0}^H(a) \leq -\sup_{i \in I} f_i(x') + \varepsilon.$$

Since ε is arbitrary, we have

$$\alpha \leq -\sup_{i \in I} f_i(x') \leq -\inf_{\langle v, x \rangle \geq 1} \sup_{i \in I} f_i(x) = (\sup_{i \in I} f_i)^H(v).$$

This is a contradiction. □

Combining Theorems 4 and 5 we get the first characterization theorem.

Theorem 6 *Let I be an arbitrary index set, and f_i be a evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for all $i \in I$, and assume the condition* (A1)*:*

(A1) $\forall x \in \mathbb{R}^n \backslash \{0\} \quad \sup_{i \in I} f_i(x) > \sup_{i \in I} f_i(0)$.

Then, for $v \in \mathbb{R}^n \backslash \{0\}, \alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$, the following conditions (i) *and* (ii) *are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \sup_{i \in I} f_i(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \langle v, x\rangle < \alpha\}$,
(ii) $\frac{v}{\alpha} \in L((\inf_{i \in I} f_i^H)^{HH}, \leq, -\beta)$.

From Theorem 3 (iii) and Theorem 5,

$$\mathrm{Hec} \bigcup_{i \in I} L(f_i^H, \leq, -\beta) \subset L((\inf_{i \in I}(f_i^H))^{HH}, \leq, -\beta) = L((\sup_{i \in I} f_i)^H, \leq, -\beta),$$

which yields the following result.

Corollary 1 *Let $v \in \mathbb{R}^n \backslash \{0\}, \alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$. If there exists $m \in \mathbb{N}, v_1, \ldots, v_m \in \mathbb{R}^n$, and $\lambda_1, \ldots, \lambda_m \in [0, \infty)$ with $\sum_{k=1}^m \lambda_k \leq 1$ such that*

$$\frac{v}{\alpha} = \sum_{k=1}^{m} \lambda_k v_k \text{ and for all } k \in \{1, \cdots, m\},\ f_i^H(v_k) \leq -\beta \text{ for some } i \in I$$

then,

$$\{x \in \mathbb{R}^n \mid \sup_{i \in I} f_i(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \langle v, x\rangle < \alpha\}.$$

Next, we show a result on the containment for the case I is arbitrary.

Theorem 7 *Let I be an arbitrary index set, and f_i be an evenly quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for all $i \in I$. Assume that the following conditions* (A1) *and* (A2) *are satisfied:*

(A1) $\forall x \in \mathbb{R}^n \backslash \{0\} \quad \sup_{i \in I} f_i(x) > \sup_{i \in I} f_i(0)$,
(A2) $\inf_{i \in I}(f_i^H)$ *is l.s.c and included in* Γ^∞.

Then for $v \in \mathbb{R}^n \backslash \{0\}, \alpha \in (0, \infty)$ and $\beta \in \mathbb{R}$, the following statements are equivalent:

(i) $L(\sup_{i \in I} f_i, <, \beta) \subset \{x \in \mathbb{R}^n \mid \langle v, x\rangle < \alpha\}$;
(ii) $\frac{v}{\alpha} \in \mathrm{Hec} L(\inf_{i \in I} f_i^H, \leq, -\beta)$.

Proof Firstly, we show that (ii) implies (i). Assume that (ii) holds. Then, from Theorem 3 (iii) and Theorem 5,

$$\mathrm{Hec} L(\inf_{i \in I} f_i^H, \leq, -\beta) \subset L((\inf_{i \in I}(f_i^H))^{HH}, \leq, -\beta) = L((\sup_{i \in I} f_i)^H, \leq, -\beta).$$

Then we have $\frac{v}{\alpha} \in L((\sup_{i \in I} f_i)^H, \leq, -\beta)$ and, by Theorem 4, (i) is derived. Next, we show (i) implies (ii). By using Theorem 4 and Theorem 5, (i) implies

$$\frac{v}{\alpha} \in L((\inf_{i \in I}(f_i^H))^{HH}, \leq, -\beta).$$

From assumption (A2) and Theorem 3, we get

$$L((\inf_{i \in I}(f_i^H))^{HH}, \leq, -\beta) = \mathrm{Hec} L(\inf_{i \in I} f_i^H, \leq, -\beta).$$

Then (ii) is satisfied. □

In the following Corollary 2, we show a set containment characterization when all f_i are quasiconvex, I is a finite set, all h_j are linear, J is an arbitrary set, and the inequalities in A and B are strict.

Corollary 2 *Let I be a finite set, J be an arbitrary set, f_i be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and included in γ^0 for each $i \in I$, $v_j \in \mathbb{R}^n \backslash \{0\}$ and $\alpha_j \in (0, \infty)$ for each $j \in J$. If condition* (A1) *holds, then the following conditions (i) and (ii) are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \forall j \in J, \langle v_j, x\rangle < \alpha_j\}$;
(ii) $\forall j \in J, \frac{v_j}{\alpha_j} \in \operatorname{Hec} \bigcup_{i \in I} L(f_i^H, \leq, -\beta)$.

Proof We can check $\bigcup_{i\in I} L(f_i^H, \leq, -\beta) = L(\inf_{i\in I} f_i^H, \leq, -\beta)$, $L(\sup_{i\in I} f_i, <, \beta) = \{x \in \mathbb{R}^n \mid \forall i \in I, f_i(x) < \beta\}$ and (A2) by using assumptions. □

6 Containment of a convex set in a reverse convex set

In this section, we present a characterization of the containment of a convex set, defined by finitely many constraints, in a reverse convex set, defined by a quasiconvex constraint. First, we introduce another concept of quasiconjugate for quasiconvex functions.

Definition 10 (*[7]*) R-quasiconjugate of f is the function $f^R : \mathbb{R}^n \to \overline{\mathbb{R}}$ such that, for any $\xi \in \mathbb{R}^n$,

$$f^R(\xi) = -\inf\{f(x) \mid \langle \xi, x\rangle \geq -1\}.$$

The R-quasiconjugate of f^R, f^{RR}, is called the R-biquasiconjugate of f.

Theorem 8 *Let $v \in \mathbb{R}^n \backslash \{0\}$, $\beta \in \mathbb{R}$. Then,*

$$L(f, <, \beta) \subset \{x \mid \langle v, x\rangle < -1\} \iff v \in L(f^R, \leq, -\beta).$$

The proof is similar to the one of Theorem 4 and so it is omitted.

Theorem 9 *Let f and h be usc quasiconvex functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Assume that $L(h, <, \alpha) \neq \emptyset$ and $0 \in L(f, <, \beta)$ for some $\alpha, \beta \in \mathbb{R}$. Then the following conditions are equivalent:*

(i) $L(f, <, \beta) \subset L(h, \geq, \alpha)$,
(ii) $0 \in L(f^H, \leq, -\beta)\backslash\{0\} + L(h^R, \leq, -\alpha)\backslash\{0\}$,
(iii) *there exists $v \in \mathbb{R}^n \backslash \{0\}$ such that $f^H(v) \leq -\beta$ and $h^R(-v) \leq -\alpha$.*

Proof It is clear that (ii) and (iii) are equivalent. Since $L(f, <, \beta)$ and $L(h, <, \alpha)$ are non-empty open convex subsets and $0 \in L(f, <, \beta)$, we have

$$\begin{aligned}
\text{(i)} \iff & L(f, <, \beta) \cap L(h, <, \alpha) = \emptyset \\
\iff & \exists v \in \mathbb{R}^n\backslash\{0\},\ \exists \gamma \in \mathbb{R} \text{ s.t.} \\
& \langle v, x\rangle > \gamma > \langle v, x'\rangle, \forall x \in L(h, <, \alpha), \forall x' \in L(f, <, \beta) \\
\iff & \exists v \in \mathbb{R}^n\backslash\{0\} \text{ s.t.} \\
& \langle v, x\rangle > 1 > \langle v, x'\rangle, \forall x \in L(h, <, \alpha), \forall x' \in L(f, <, \beta) \\
\iff & \exists v \in \mathbb{R}^n\backslash\{0\} \text{ s.t. } f^H(v) \leq -\beta \text{ and } h^R(-v) \leq -\alpha,
\end{aligned}$$

by the separation theorem. □

Substituting $\sup_{i\in I} f_i$ into f, we obtain the following theorem.

Theorem 10 *Let I be a finite set, f_i be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ for each $i \in I$, and h be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$. Assume that $L(h, <, \alpha) \neq \emptyset$ and $\sup_{i\in I} f_i(0) < \beta$ for some $\alpha, \beta \in \mathbb{R}$. Then the following conditions are equivalent:*

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I,\ f_i(x) < \beta\} \subset L(h, \geq, \alpha)$,
(ii) $0 \in L((\sup_{i\in I} f_i)^H, \leq, -\beta)\backslash\{0\} + L(h^R, \leq, -\alpha)\backslash\{0\}$,
(iii) *there exists* $v \in \mathbb{R}^n\backslash\{0\}$ *such that* $(\sup_{i\in I} f_i)^H(v) \leq -\beta$ *and* $h^R(-v) \leq -\alpha$.

Proof Since I is a finite set, we have $\sup_{i\in I} f_i$ is usc and $L(\sup_{i\in I} f_i, \leq, \beta) = \{x \in \mathbb{R}^n \mid \forall i \in I,\ f_i(x) < \beta\}$. By using Theorem 9, we conclude the proof. □

The next corollary characterizes the set containment in the case that all f_i are quasiconvex, I is a finite set, all h_j are quasiconcave, J is an arbitrary set, the inequalities in A are strict, and the inequalities in B are weak.

Corollary 3 *Let I be a finite set, J be an arbitrary set, f_i be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ included in γ^0 for each $i \in I$, h_j be an usc quasiconvex function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$ and $\alpha_j \in (0, \infty)$ for each $j \in J$. Assume that*

(A1) $\forall x \in \mathbb{R}^n\backslash\{0\} \quad \sup_{i\in I} f_i(x) > \sup_{i\in I} f_i(0)$,

and $L(h_j, <, \alpha_j) \neq \emptyset$ for each $j \in J$ and $\sup_{i\in I} f_i(0) < \beta$ for some $\beta \in \mathbb{R}$. Then the following conditions are equivalent:

(i) $\{x \in \mathbb{R}^n \mid \forall i \in I,\ f_i(x) < \beta\} \subset \{x \in \mathbb{R}^n \mid \forall j \in J, h_j(x) \geq \alpha_j\}$,
(ii) *for each* $j \in J$, $0 \in \operatorname{Hec} \bigcup_{i\in I} L(f_i^H, \leq, -\beta)\backslash\{0\} + L(h_j^R, \leq, -\alpha_j)\backslash\{0\}$,
(iii) *for each* $j \in J$, *there exists* $v \in \mathbb{R}^n\backslash\{0\}$ *such that*

$$v \in \operatorname{Hec} \bigcup_{i\in I} L(f_i^H, \leq, -\beta) \text{ and } h_j^R(-v) \leq -\alpha_j.$$

7 Discussion

In this section, we compare the main results in this paper with previous ones in Refs. 1, 2, 4 and 5. Consider the sets

$$F = \{x \in \mathbb{R}^n \mid f_t(x) \leq 0, \forall t \in W; g_s(x) < 0, \forall s \in S; l_e(x) = 0, \forall e \in E\},$$

and

$$G = \{x \in \mathbb{R}^n \mid k_i(x) < 0, \forall i \in I; h_j(x) \leq 0, \forall j \in J\},$$

where W, S, E, I and J are pairwise disjoint sets, $W \cup S \cup E \neq \emptyset, I \cup J \neq \emptyset$, and $\{f_t, t \in W\}, \{g_s, s \in S\}, \{l_e, e \in E\}, \{k_i, i \in I\}$ and $\{h_j, j \in J\}$ are functions from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$.

We summarize in Table 1 the results on set containments in the similar way in [1]. No. from 1 to 7 are previous results, No. 8 and 9 are our results in this paper. The columns 3, 4, 5, 6 and 7 inform on the cardinality of the index sets, which can be empty, finite or arbitrary (abbreviated as "∅", "Fin" and "Arb", respectively), and the columns 8, 9, 10, 11 and 12 inform about assumptions of functions, which can be affine, quadratic concave, differentiable convex, differentiable concave, convex, concave, quasiconvex and quasiconcave

Table 1 Literature on set containments

No.	Ref.	W	S	E	I	J	$\{f_t\}$	$\{g_s\}$	$\{l_e\}$	$\{k_i\}$	$\{h_j\}$
1	[5]	Fin	∅	∅	∅	Fin	Aff	–	–	–	Aff
2	[5]	Fin	∅	∅	∅	Fin	Aff	–	–	–	Quad
3	[5]	Fin	∅	∅	∅	Fin	Dconv	–	–	–	Dconc
4	[4]	Arb	∅	∅	∅	Fin	Conv	–	–	–	Aff
5	[4]	Arb	∅	∅	∅	Fin	Conv	–	–	–	Conc
6	[1]	Arb	***	Arb	Arb	***	Aff	Aff	Aff	Aff	Aff
7	[2]	Arb	Arb	∅	∅	Fin	Conv	Conv	–	–	Conc
8	–	∅	Fin	∅	Fin	∅	Qconv	–	–	–	Aff
9	–	∅	Fin	∅	∅	Fin	Qconv	–	–	–	Qconc

(abbreviated as “Aff”, “Quad”, “Dconv”, “Dconc”, “Conv”, “Conc”, “Qconv” and “Qconc”, respectively). “***” means that $S \cup J \neq \emptyset$.

In the rest of the section, we compare No. 4 with No. 8 especially. Section 5 characterizes the containment in the form

$$L(f, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\},$$

whereas Jeyakumar [4] considered inclusions of the form

$$L(f, \leq, \beta) \subset \{x \mid \langle v, x\rangle \leq \alpha\}.$$

We discuss conditions guaranteeing the equivalence of both inclusions. It is easy to show that for any $v \in \mathbb{R}^n \backslash \{0\}$,

$$\mathrm{int}\{x \mid \langle v, x\rangle \leq \alpha\} = \{x \mid \langle v, x\rangle < \alpha\}, \quad \mathrm{cl}\{x \mid \langle v, x\rangle < \alpha\} = \{x \mid \langle v, x\rangle \leq \alpha\}.$$

Moreover, if f is continuous, we have

$$L(f, <, \beta) \subset \mathrm{int} L(f, \leq, \beta), \quad \mathrm{cl} L(f, <, \beta) \subset L(f, \leq, \beta)$$

are satisfied, but the converse inclusions are not true in general. When the equalities are fulfilled in these inclusions, we can show easily that our form and Jeyakumar’s form are equivalent. For our purpose, we show the following lemmas:

Lemma 1 *Let* $A, B \subset \mathbb{R}^n$. *If* $\mathrm{int} A = \emptyset$ *and* $\mathrm{int}(\mathrm{cl} B) = \mathrm{int} B$, *then we have* $\mathrm{int}(A \cup B) = \mathrm{int} B$.

Proof Inclusion $\mathrm{int}(A \cup B) \supset \mathrm{int} B$ is obvious. Conversely, for any $x \in \mathrm{int}(A \cup B)$, there exists $r > 0$ satisfying $B(x, r) \subset A \cup B$. If $\mathrm{int}(B(x, r) \cap B^c) \neq \emptyset$, then we have a contradiction since $\mathrm{int} A = \emptyset$ and $B(x, r) \cap B^c \subset (A \cup B) \cap B^c \subset A$ hold. Therefore

$$\emptyset = \mathrm{int}(B(x, r) \cap B^c) = B(x, r) \cap \mathrm{int}(B^c) = B(x, r) \cap (\mathrm{cl} B)^c,$$

and then $B(x, r) \subset \mathrm{cl} B$. By using assumption $\mathrm{int}(\mathrm{cl} B) = \mathrm{int} B$, we obtain $B(x, r) \subset \mathrm{int}(\mathrm{cl} B) = \mathrm{int} B \subset B$. This shows that $x \in \mathrm{int} B$. □

Lemma 2 *Let* $A, B \subset \mathbb{R}^n$. *If* $\mathrm{int} A = \emptyset$ *and* B *is a convex set with* $\mathrm{int} B \neq \emptyset$, *then we have*

(i) $\mathrm{int}(A \cup B) = \mathrm{int} B$,
(ii) $\mathrm{int}(A \cup B^c) = \mathrm{int} B^c$.

Proof Since B is convex and $\text{int}B \neq \emptyset$, we have

$$\text{int}(\text{cl}B) = \text{int}B \quad \text{and} \quad \text{cl}(\text{int}B) = \text{cl}B.$$

The second equation yields $\text{int}(\text{cl}(B^c)) = \text{int}(B^c)$. Therefore (i) and (ii) are proved by using Lemma 1. □

Theorem 11 *Let f be a continuous quasiconvex function from $\mathbb{R}^n$ to $\mathbb{R}$, $v \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$. If $\text{int}L(f, =, \beta) = \emptyset$ and $\text{int}L(f, <, \beta) \neq \emptyset$ for some $\beta \in \mathbb{R}$, then we have*

(i) $L(f, <, \beta) = \text{int}L(f, \leq, \beta)$,
(ii) $\text{cl}L(f, <, \beta) = L(f, \leq, \beta)$. *Moreover*

$$L(f, <, \beta) \subset \{x \mid \langle v, x\rangle < \alpha\} \iff L(f, \leq, \beta) \subset \{x \mid \langle v, x\rangle \leq \alpha\}.$$

Proof (i) Put $A = L(f, =, \beta)$ and $B = L(f, <, \beta)$. By using Lemma 2 (i), $\text{int}L(f, \leq, \beta) = \text{int}L(f, <, \beta) = L(f, <, \beta)$ because f is usc. Next we show (ii). Put $A = L(f, =, \beta)$ and $B = L(f, \leq, \beta)$. By using the Lemma 2 (ii), we have $\text{int}L(f, \geq, \beta) = \text{int}L(f, >, \beta)$, and equivalently $\text{cl}L(f, <, \beta) = \text{cl}L(f, \leq, \beta)$. Since f is lsc, $\text{cl}L(f, \leq, \beta) = L(f, \leq, \beta)$. The equivalence is straightforward consequence of statements (i) and (ii). □

Remark 1 If every f_i is convex and $\text{dom}(\sup_{i \in I} f_i) = \mathbb{R}^n$ and condition [A1] holds, then the assumptions of Theorem 6 is satisfied. Also if $\inf_{x \in \mathbb{R}^n} \sup_{i \in I} f_i(x) < 0$, we can check that $\text{int}L(\sup_{i \in I} f_i, =, 0) = \emptyset$ and $\text{int}L(\sup_{i \in I} f_i, <, 0) \neq \emptyset$. For any $\alpha \in (0, \infty)$ and any $v \in \mathbb{R}^n \setminus \{0\}$, we have the following characterization concerned with the Fenchel conjugate and H-quasiconjugate :

$$(v, \alpha) \in \text{cl}\left(\text{coneco} \bigcup_{i \in I} \text{epi} f_i^*\right) \iff \frac{v}{\alpha} \in L((\inf_{i \in I} f_i^H)^{HH}, \leq, 0).$$

Acknowledgements The authors are grateful to the anonymous refree for careful reading of the manuscript and many comments and suggestions improved the quality of the paper.

References

1. Goberna, M.A., Rodríguez, M.M.L.: Analyzing linear systems containing strict inequalities via evenly convex hulls. Eur. J. Oper. Res. **169**, 1079–1095 (2006)
2. Goberna, M.A., Jeyakumar, V., Dinh, N.: Dual characterizations of set containments with strict convex inequalities. J. Global Optim. **34**, 33–54 (2006)
3. Greenberg, H.J., Pierskalla, W.P.: Quasi-conjugate functions and surrogate duality. Cah. Cent. Étud. Rech. Opér. **15**, 437–448 (1973)
4. Jeyakumar, V.: Characterizing set containments involving infinite convex constraints and reverse-convex constraints. SIAM J. Optim. **13**, 947–959 (2003)
5. Mangasarian, O.L.: Set containment characterization. J. Global Optim. **24**, 473–480 (2002)
6. Thach, P.T.: Quasiconjugates of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint, and applications. J. Math. Anal. Appl. **159**, 299–322 (1991)
7. Thach, P.T.: Diewert-Crouzeix conjugation for general quasiconvex duality and applications. J. Optim. Theory Appl. **86**, 719–743 (1995)